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# Time-dependent Variational Inequalities and Applications to Equilibrium Problems

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**Abstract.** We present two different applications of time-dependent variational inequalities. First we propose a new model of time-dependent distributed markets networks which includes delay effects. Afterwards, we deal with time-dependent and elastic models of transportation networks.

**Key words:** Delay, Economic Networks, Variational Inequalities, Quasi-variational inequalities

## 1. Introduction

Time is central in our reality, in the physical-technological world as well as in the socio-economic world, that is why in the last years several authors have dealt with time-dependent problems in the framework of variational inequalities, which express equilibrium conditions. Very recently *time* has been introduced in variational models describing traffic problems as well as equilibrium markets problems. In this note we present two generalizations of time-dependent variational inequalities. The first topic concerns *delay effects* in economic networks. Delay effects are well known in mathematical theory of population dynamics and in mathematical modelling of many physical phenomena [8], [5]. Variational Inequalities with delay have recently been considered in time dependent traffic models [11]. In economic markets models the importance of delay derives from the fact that, even if the propagation of information through the network could be considered in certain cases instantaneous, producers and consumers usually take their decisions after they get all the data available. Mathematical modelling of economic markets by network theory has a long history [1], [10]. In particular we are interested in spatial price equilibrium problems, where one wants to compute the commodity supply prices, demand prices and trade flows that satisfy a certain equilibrium condition. Our approach extends previous works [9] and [2]. The second application concerns equilibrium problems in transportation networks. In this case, variational analysis is exploited to the aim of surveying time-dependent and elastic models, when capacity constraints are introduced on all the routes. The elasticity of travel demands requires a formulation in terms of quasi-variational inequalities.

Therefore, by applying some recent results [3], [4], [7], we provide a theorem for the existence of solutions and give an example.

## 2. Delay Effects

Before introducing our problem, let us briefly review the time-dependent model proposed in [2], where the authors consider a single commodity which is produced in $n$ supply markets and sold in $m$ demand markets, which are spatially separated. Markets are considered $\forall t \in [0, T]$. For each $t$ there is a a total supply vector $g(t) \in \Re^n$, the supply price vector $p(t) \in \Re^n$, the total demand vector $f(t) \in \Re^m$, the demand price vector $q(t) \in \Re^m$, the flow of commodity vector $x(t) \in \Re^{nm}$ and the unitary transportation cost vector $\pi[x(t)] \in \Re^{nm}$. Feasible vectors $u(t) = [p(t), q(t), x(t)]$ must satisfy, almost everywhere on $[0, T]$, the constraints:

$$u(t) \in \Pi_{i=1}^{n} [\underline{p}_i(t), \infty[ \times \Pi_{j=1}^{m} [0, \overline{q}_j(t)] \times \Pi_{i=1}^{n} \Pi_{j=1}^{m} [0, \overline{x}_{ij}(t)]$$

where $\underline{p}_i(t), \overline{q}_j(t), \overline{x}_{ij}(t)$ are given.

$$u \in L^2([0, T], \Re^n) \times L^2([0, T], \Re^m) \times L^2([0, T], \Re^{nm}) \equiv L \equiv L_1 \times L_2 \times L_3.$$

The feasible set is then given by:

$$K \equiv K_1 \times K_2 \times K_3 \equiv$$

$$\{p \in L_1 : \underline{p}(t) \leqslant p(t), \text{a.e.} t \in [0, T]\} \times$$

$$\{q \in L_2 : 0 \leqslant q(t) \leqslant \overline{q}(t), \text{a.e.} t \in [0, T]\} \times$$

$$\{x \in L_3 : 0 \leqslant x(t) \leqslant \overline{x}(t), \text{a.e.} t \in [0, T]\}.$$

The following operators are then given:

$$g : K_1 \mapsto L_1, \; f : K_2 \mapsto L_2, \; \pi : K_3 \mapsto L_3$$

which to each trajectory $p \in K_1$, $q \in K_2$ and $x \in K_3$ associate the supply $g$, the demand $f$ and the cost $\pi$, respectively.

Introducing the excess supply $s_i(t)$ and the excess demand $t_j(t)$ we must have:

$$g_i[p(t)] = \Sigma_{j=1}^{m} x_{ij}(t) + s_i(t) \quad i = 1 \ldots n \tag{2.1}$$

$$f_j[q(t)] = \Sigma_{i=1}^{n} x_{ij}(t) + t_j(t) \quad j = 1 \ldots m \tag{2.2}$$

where $s \in L_1$ and $t \in L_2$.

DEFINITION 1. $u^* = (p^*, q^*, x^*) \in K$ *is a Dynamical Equilibrium iff* $\forall i = 1.., n$, $\forall j = 1.., m$, *and a.e.* $t \in [0, T]$ *we have:*

$$s_i^*(t) > 0 \longrightarrow p_i^*(t) = \underline{p}_i(t), \; p_i^*(t) > \underline{p}_i(t) \longrightarrow s_i^*(t) = 0 \tag{2.3}$$

$$t_j^*(t) > 0 \longrightarrow q_j^*(t) = \overline{q}_j(t), \; q_j^*(t) < \overline{q}_j(t) \longrightarrow t_j^*(t) = 0 \tag{2.4}$$

$$p_i^*(t) + \pi_{ij}[x^*(t)] : \begin{cases} \geqslant q_j^*(t) \text{ if } x_{ij}^*(t) = 0 \\ = q_j^*(t) \text{ if } 0 < x_{ij}^*(t) < \overline{x}_{ij}(t) \\ \leqslant q_j^*(t) \text{ if } x_{ij}^*(t) = \overline{x}_{ij}(t). \end{cases} \tag{2.5}$$

Now we want to introduce delay effects in the previous model. In our approach, we suppose that the excesses of supply and demand are quantities which are observed in the interval $[0, T]$ and that prices are adjusted as a consequence of this observation. If the observations are made at time $t$ we require that new prices are established at the time $t+h$. In particular, we give the new definition of retarded equilibrium which implies that: if an excess in the production is observed a time $t$, then the supply market will plan a lower supply price at time $t+h$; if the demand market does not receive, at time $t$, the whole quantity of commodity that it asks for, then it will plan a higher demand price at time $t+h$; We shall retain the previous notation, the only difference being that now all the functions are defined in $[0, T+h]$, $h > 0$. Thus, the functional space for the trajectories is:

$$L^2([0, T+h], \Re^n) \times L^2([0, T+h], \Re^m) \times L^2([0, T+h], \Re^{nm})$$

and the feasible set is given by:

$$\begin{aligned} &K \equiv K_1 \times K_2 \times K_3 = \\ &\{p \in L_1 : \underline{p}(t) \leqslant p(t), \text{a.e.} t \in [0, T+h]\} \times \\ &\{q \in L_2 : 0 \leqslant q(t) \leqslant \overline{q}(t), \text{a.e.} t \in [0, T+h]\} \times \\ &\{x \in L_3 : 0 \leqslant x(t) \leqslant \overline{x}(t), \text{a.e.} t \in [0, T+h]\}. \end{aligned}$$

The following operators are then given:

$$g : K_1 \mapsto L_1, \; f : K_2 \mapsto L_2, \; \pi : K_3 \mapsto L_3$$

which to each trajectory $p \in K_1$, $q \in K_2$ and $x \in K_3$ associate the supply $g$, the demand $f$ and the cost $\pi$, respectively.

Introducing the excess supply $s_i(t)$ and the excess demand $t_j(t)$ we must have, almost everywhere in $[0, T]$:

$$g_i[p(t)] = \Sigma_{j=1}^m x_{ij}(t) + s_i(t) \;\; i = 1...n \tag{2.6}$$

$$f_j[q(t)] = \Sigma_{i=1}^n x_{ij}(t) + t_j(t) \quad j = 1...m \tag{2.7}$$

where $s \in L_1$ and $t \in L_2$.

Now we can give the new definition of Equilibrium which takes into account the delay:

DEFINITION 2. *$u^* = (p^*, q^*, x^*) \in K$ is a Retarded Dynamical Equilibrium iff $\forall i = 1.., n$, $\forall j = 1.., m$, and a.e. $t \in [0, T]$ we have:*

$$s_i^*(t) > 0 \longrightarrow p_i^*(t+h) = \underline{p}_i(t+h), \;\; p_i^*(t+h) > \underline{p}_i(t+h) \longrightarrow s_i^*(t) = 0 \tag{2.8}$$

$$t_j^*(t) > 0 \longrightarrow q_j^*(t+h) = \overline{q}_j(t+h), \;\; q_j^*(t+h) < \overline{q}_j(t+h) \longrightarrow t_j^*(t) = 0 \tag{2.9}$$

$$p_i^*(t) + \pi_{ij}[x^*(t)] : \begin{cases} \geqslant q_j^*(t) \text{ if } x_{ij}^*(t) = 0 \\ = q_j^*(t) \text{ if } 0 < x_{ij}^*(t) < \overline{x}_{ij}(t) \\ \leqslant q_j^*(t) \text{ if } x_{ij}^*(t) = \overline{x}_{ij}(t). \end{cases} \tag{2.10}$$

Let us prove that if a vector $u^*$ is a Retarded Equilibrium, then it satisfies a variational inequality. The opposite implication will be proved under additional conditions.

Thus, if we assume first that (2.8) holds then we can show that:

$$[p_i(t+h) - p_i^*(t+h)][s_i^*(t)] \geqslant 0 \text{ a.e.} t \in [0, T]. \tag{2.11}$$

In fact, if for a fixed $t$ (outside the null set where (2.8) is not valid) $p_i^*(t+h) = \underline{p}(t+h)$, from (2.8) it follows that $s_i^*(t) = g_i[p^*(t)] - \Sigma_j x_{ij}(t) > 0$ so that the product (2.11) is $\geqslant 0$.

If $p_i^*(t+h) > \underline{p}(t+h)$ it follows from (2.8) that $s_i^*(t) = 0$.

Then, summing on $i$, as the finite union of null sets is a null set, one gets:

$$\Sigma_i \{g_i[p^*(t)] - \Sigma_j x_{ij}^*(t)\}\{p_i(t+h) - p_i^*(t+h)\} \geqslant 0 \text{ a.e.on} [0, T].$$

Analogously one gets:

$$-\Sigma_j \{f_j[q^*(t)] - \Sigma_i x_{ij}^*(t)\}\{q_j(t+h) - q_j^*(t+h)\} \geqslant 0 \text{ a.e.on} [0, T].$$

Assume now the validity of (2.10) and study for a fixed $t$, (outside the null set where (2.10) is not valid), the sign of the quantity:

$$\{p_i^*(t) + \pi_{ij}[x^*(t)] - q_j^*(t)\}\{x_{ij}(t) - x_{ij}^*(t)\}. \tag{2.12}$$

We must consider three cases:

(i) if $x_{ij}^*(t)=0$, then $p_i^*(t)+\pi_{ij}[x^*(t)]\geqslant q_j^*(t)$ so that the first factor in the product (2.12) is nonnegative and, because flows are nonnegative, the whole product is nonnegative.
(ii) if $0<x_{ij}^*(t)<\overline{x}_{ij}(t)$ then $p_i^*(t)+\pi_{ij}[x^*(t)]=q_j^*(t)$, so that the product is zero.
(iii) if $x_{ij}^*(t)=\overline{x}_{ij}(t)$, both the factors are nonpositive so that the product is nonnegative.

Summing up on $i$ and $j$, and integrating on $[0,T]$, we get finally the following Variational Inequality:

$$\int_0^T \{\Sigma_i(g_i[p^*(t)]-\Sigma_j x_{ij}^*(t))(p_i(t+h)-p_i^*(t+h))+$$
$$-\Sigma_j(f_j[q^*(t)]-\Sigma_i x_{ij}^*(t))(q_j(t+h)-q_j^*(t+h))+$$
$$+\Sigma_i\Sigma_j(p_i^*(t)+\pi_{ij}[x^*(t)]-q_j^*(t))(x_{ij}(t)-x_{ij}^*(t))\}dt\geqslant 0 \ \forall u\in K. \tag{2.13}$$

The Variational Inequality just introduced implies the equilibrium conditions (2.6)–(2.10) under natural, additional, assumptions as the following theorem shows:

THEOREM 1. *Let the variational inequality* (14) *hold and moreover*:

(1) $x_{ij}(t)>0$ *on* $E\subset[0,T]$, $(|E|>0)$, $\longrightarrow \pi_{ij}[x(t)]>0$ *on* $E$.
(2) *if* $q_j(t)>0$ *on* $E\subset[0,h]$ *then* $t_j(t)\geqslant 0$ *on* $E$.

*Then the equilibrium conditions* (2.6)–(2.10) *hold.*

*Proof.* We shall prove (2.6), (2.8), (2.10), (2.7), (2.9) In fact, putting in (14): $q=q^*$ and $x=x^*$, or $p=p^*$ and $x=x^*$ or $p=p^*$ and $q=q^*$ we get, respectively:

$$\int_0^T \Sigma_i(g_i[p^*(t)]-\Sigma_j x_{ij}^*(t))(p_i(t+h)-p_i^*(t+h))dt\geqslant 0\ \forall p\in K_1 \tag{2.14}$$

$$-\int_0^T \Sigma_j(f_j[q^*(t)]-\Sigma_i x_{ij}^*(t))(q_j(t+h)-q_j^*(t+h))dt\geqslant 0\ \forall q\in K_2 \tag{2.15}$$

$$\int_0^T \Sigma_i\Sigma_j(p_i^*(t)+\pi_{ij}[x^*(t)]-q_j^*(t))(x_{ij}(t)-x_{ij}^*(t))dt\geqslant 0 \ \forall x\in K_3. \tag{2.16}$$

In order to show that from these relations one gets the equilibrium conditions (2.6)–(2.10) let us suppose first that (2.6) does not hold, (with $s_i(t) \geqslant 0$ a.e. on $[0,T]$). Then $\exists i'$ and $\exists E \subset [0,T]$ with $|E|>0$ such that:

$$s^*_{i'}(t) = g_{i'}[p^*(t)] - \Sigma_j x^*_{i'j}(t) < 0 \text{ on E.}$$

Then let us choose: $p_i(t) = p^*_i(t)$ if $i \neq i'$

$$p_{i'}(t): \begin{cases} = p^*_{i'}(t) & \text{if } t \in [h, T+h] \backslash E + h \\ > p^*_{i'}(t) & \text{if } t \in E + h \\ \forall & \text{if } t \in [0,h] \end{cases}$$

which implies:

$$p_{i'}(t+h): \begin{cases} = p^*_{i'}(t+h) & \text{if } t \in [0,T] \backslash E \\ > p^*_{i'}(t+h) & \text{if } t \in E \\ \forall & \text{if } t \in [-h,0] \end{cases}$$

with this choice we get, in (15), the contradiction:

$$\int_0^T \Sigma_i (g_i[p^*(t)] - \Sigma_j x^*_{ij}(t))(p_i(t+h) - p^*_i(t+h))\mathrm{d}t =$$

$$\int_E (g_{i'}[p^*(t)] - \Sigma_j x^*_{i'j}(t))(p_{i'}(t+h) - p^*_{i'}(t+h))\mathrm{d}t < 0.$$

In order to prove (2.7), with $t^*_j(t) \geqslant 0$ a.e. on $[0,T]$, consider two cases.

(i) if $q^*_j(t+h) \in ]0, \overline{q}_j(t+h)]$, by choosing $q_j(t+h) = 0$ it follows, in (16) $t_j(t) \geqslant 0$

(ii) if it exists an index $j'$: $q^*_{j'}(t+h) = 0$, on $E \subset [0,T]$, $|E|>0$ we can prove that $t^*_j(t) \geqslant 0$ on $E$. In fact, let us suppose that $t^*_j(t) < 0$ on a set of positive measure $E' \subset E$, so that:

$$f'_j[q^*(t)] - \Sigma_i x^*_{ij'}(t) < 0, \text{ on } E'.$$

Then it exists an index $i \in \{1,2,\ldots,n\} : x^*_{ij'}(t) > 0$. It follows:

$$p^*_i(t) + \pi_{ij'}[x^*(t)] \leqslant q'_j(t) = 0$$

if $E' \cap [h, T+h] \neq \phi$ we get the contradiction $\pi_{ij'}[x^*(t)] \leqslant 0$.

If, on the contrary, $E' \subset [0,h]$, from our second hypothesis it follows that $t'_j(t) \geqslant 0$.

The other proofs are similar.

## 3. An Example

Let us consider the case of two producer markets and one consumer, with the following data:

$$g_1[p(t)] = a_1 p_1(t) + b(t);\ g_2[p(t)] = a_2 p_2(t)$$

$$f[q(t)] = cq(t) + d(t)$$

$$\pi_{11}[x(t)] = c_1 x_{11}(t);\ \pi_{21}[x(t)] = c_2 x_{21}(t).$$

Let us recall that the excess of production and of demand satisfy:

$$g_1[p(t)] = x_{11}(t) + s_1(t);\ g_2[p(t)] = x_{21}(t) + s_2(t)$$

$$f[q(t)] = x_{11}(t) + x_{21}(t) + t(t)$$

$a_1, a_2, c, c_1, c_2, b(t), d(t)$ are nonnegative parameters.

The feasible set $K$ is defined by the following constraints:

$$p_1 \geqslant 0, p_2 \geqslant 0, 0 \leqslant q(t) \leqslant Q(t), x_{11} \geqslant 0, x_{21} \geqslant 0.$$

Then one can verify that the variational inequality has the following solution in the interval $[h, T]$:

$$p_1^*(t) = 0$$

$$x_{11}^*(t) = \frac{\mathrm{d}(t)(1 + a_2 c_2)}{(1 - cc_1)(1 + a_2 c_2) + a_2 c_1}$$

$$x_{21}^*(t) = \frac{a_2 c_1}{1 + a_2 c_2} \frac{\mathrm{d}(t)(1 + a_2 c_2)}{(1 - cc_1)(1 + a_2 c_2) + a_2 c_1}$$

$$p_2^*(t) = \frac{c_1}{1 + a_2 c_2} \frac{\mathrm{d}(t)(1 + a_2 c_2}{(1 - cc_1)(1 + a_2 c_2) + a_2 c_1}$$

$$q^*(t) = \frac{\mathrm{d}(t) c_1 (1 + a_2 c_2)}{(1 - cc_1)(1 + a_2 c_2) + a_2 c_1}.$$

The feasibility conditions are:

$$cc_1 < 1$$

$$\mathrm{d}(t) \leqslant Q(t)\frac{(1-cc_1)(1+a_2c_2)+a_2c_1}{c_1(1+a_2c_2)}$$

$$d(t) < b(t)\frac{(1-cc_1)(1+a_2c_2)+a_2c_1}{(1+a_2c_2)}.$$

In the interval $[0,h]$ we have the following solution:

$$p_2^* = \frac{c_1 d(t)+p_1^*(t)}{D}$$

$$x_{11}(t) = \frac{1}{D}\{p_1^*(t)[c(1+c_2a_2)-a_2]+\mathrm{d}(t)(1+a_2c_2)\}$$

$$x_{21}^*(t) = a_2\frac{c_1 d(t)+p_1^*(t)}{D}$$

$$q^* = (1+c_2a_2)a_2\frac{c_1\mathrm{d}(t)+p_1^*(t)}{D}$$

$p_1^*(t)$ can be chosen arbitrarily provided it satisfies the feasibility conditions which are:

$$cc_2 > 1;\ c(1+c_1) < 1$$

$$q^*(t) \leqslant Q(t)$$

$$\mathrm{d}(t) < \frac{(1+c_2a_2)(1-c_1c)+1+c_1a_2}{1+c_2a_2}b(t).$$

Let us fix numerical values for the parameters: $h=1$, $T=2$, $c_1=1$, $c_2=10$, $c=1/5, a_1=a_2=1$. In the interval $[1,3]$ let $Q(t)=b(t)=165t/49$, $d(t)=t$. Then one can verify that the solution is:

$$x_{11}^*(t)=55t/49, x_{21}^*(t)=55t/539, p_1^*(t)=0, p_2^*(t)=55t/539, q^*(t)=55t/49.$$

In the interval $[0,1]$, let us fix $Q(t)=55t/27$, $d(t)=t$, then the solution is:

$$\begin{aligned} x_{11}^*(t) &= (5/54)(6p_1^*(t)+11t), x_{21}^*(t)=(5/54)(t+p_1^*(t)p_2^* \\ &= (5/54)(t+p_1^*(t))q^*=(55/54)(t+p_1^*(t)) \end{aligned}$$

where $p_1^*(t)$ can be chosen arbitrarily.

## 4. The Traffic Equilibrium Problem: Time-dependent and Elastic Cases

Our purpose is to present another class of equilibrium problems which can be examined by putting to use quasi-variational inequalities. In fact, we consider time-dependent and elastic models of transportation networks and discuss the existence of equilibria. We also impose some capacity restrictions on the routes. It is worth noting that the time-dependent formulation is required when data evolve in time, whereas when travel demands depend on the equilibrium distribution the elastic framework is necessary.

Let us consider a traffic network where:
$W$ is the set of Origin Destination (O/D) pairs $w_j$, $j=1,2,\dots,l$;
$\mathcal{R}_j$ is the set of routes $R_r$, $r=1,2,\dots,m$, which connect the pair $w_j$;
$\Phi$ is the incidence matrix, whose elements are

$$\varphi_{jr}=\begin{cases}1 \text{ if } R_r\in\mathcal{R}_j\\ 0 \text{ otherwise.}\end{cases}$$

For technical reason, the functional setting is the reflexive Banach space $L^2(0,T;\mathbb{R}^m_+)$.

Let us assume that:

(a) $C:[0,T]\times\mathbb{R}^m_+\to\mathbb{R}^m_+$ is the route cost function;
(b) $\rho:[0,T]\times\mathbb{R}^m_+\to\mathbb{R}^l_+$ is the elastic demand, depending on the equilibrium pattern;
(c) $C(t,v)$ is measurable on $t$ $\forall v\in\mathbb{R}^m_+$, continuous on $v$ for $t$ a.e. in $[0,T]$,

$$\exists\gamma\in L^2(0,T):|C(t,v)|\leqslant\gamma(t)+|v|;$$

(d) $\rho(t,v)$ is measurable on $t$ $\forall v\in\mathbb{R}^m_+$, continuous on $v$ for $t$ a.e. in $[0,T]$,

$$\exists\psi\in L^1(0,T):|\rho(t,v)|\leqslant\psi(t)+|v|^2;$$

(e) $\exists h(t)\geqslant 0$ a.e. in $[0,T]$, $h\in L^2(0,T)$:

$$\forall v_1,v_2\in\mathbb{R}^m_+,\ |\rho(t,v_1)-\rho(t,v_1)|\leqslant h(t)|v_1-v_2|$$

(f) $\lambda(t),\mu(t)\in L^2(0,T;\mathbb{R}^m_+)$, $\lambda(t)\leqslant\mu(t)$ a.e. in $[0,T]$ are the capacity restrictions.

Then if $E$ is a non-empty, compact, convex subset of $L^2(0,T;\mathbb{R}^m_+)$, the set of feasible flows is the set-valued function defined as follows:

$$K:E\to 2^{L^2(0,T;\mathbb{R}^m_+)};$$

$$\begin{aligned}K(H)=\Big\{&F\in L^2(0,T;\mathbb{R}^m_+):\lambda_r(t)\leqslant F_r(t)\leqslant\mu_r(t) \text{ a.e. in } [0,T],\\ &r=1,2,\dots,m;\sum_{r=1}^m\varphi_{jr}F_r(t)=\frac{1}{T}\int_0^T\rho_j(t,H(\tau))\mathrm{d}\tau\\ &\text{a.e. in } [0,T], j=1,2,\dots,l\Big\}.\end{aligned}$$

In order to ensure the nonvoidness of $K(H)$, we suppose that $\Phi\lambda(t) \leqslant \Phi F(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$. We consider a formulation of the equilibrium problems where the dependence of the flows $F_r$ on the unknown solution $H$ is assumed on average with respect to time, *i.e.*, $\sum_{r=1}^{m} \varphi_{jr} F_r(t) = \frac{1}{T}\int_0^T \rho_j(t, H(\tau))\mathrm{d}\tau$, see [3, 7] for more details. In conclusion, the elastic and time-dependent equilibrium problem is expressed by the following quasi-variational inequality:
"Find $H \in K(H)$:

$$\int_0^T C(t,H(t))(F(t)-H(t))\mathrm{d}t \geqslant 0, \qquad \text{a.e. in } [0,T] \quad \forall F \in K(H)\text{''}. \tag{4.17}$$

Regarding the existence of solutions, let us recall the following general result (see [13]):

THEOREM 2. *Let $X$ be a locally convex, Hausdorff topological vector space, $E$ a nonempty compact, convex subset of $X$, $C: E \to X^*$ a continuous function, $K: E \to 2^E$ a closed lower semicontinuous multifunction with $K(H) \subset E$ nonempty, compact, convex $\forall H \in E$. Then, there exists a solution for the quasi-variational inequality*:

$$H \in K(H),\ \langle F-H, C(H)\rangle \geqslant 0 \quad \forall F \in K(H).$$

Theorem 2 enables us to prove the following existence result related to our equilibrium problem (see [12]):

THEOREM 3. *Under the assumptions* $(a)$, $(b)$, $(c)$, $(d)$, $(e)$ *and* $(f)$ *on $C$ and $\rho$, and if $K(H) \subset E$, $\forall H \in E$, then the quasi-variational inequality* (4.17) *admits a solution.*

*Proof.* At first we can observe that under the hypotheses (a), (b) and if $H(t) \in L^2(0,T;\mathbb{R}^m_+)$, it results that

$$C(t,H(t)) \in L^2(0,T;\mathbb{R}^m_+) \text{ and } \rho(t,H(t)) \in L^1(0,T;\mathbb{R}^l_+).$$

Moreover, by a) and b) it follows that $C$ and $\rho$ belong to the class of Nemytskii operators, therefore if $\{H^n\} \longrightarrow^{L^2} H$ then

$$\|C(t,H^n(t)) - C(t,H(t))\|_{L^2} \to 0,\ \|\rho(t,H^n(t)) - \rho(t,H(t))\|_{L^1} \to 0$$

and the functions $C, \rho$ are $L^2$ and $L^1$-continuous respectively.
In order to prove that $K$ is a closed multifunction, we show that:

$$\forall \{H^n\} \longrightarrow^{L^2} H,\ \forall \{F^n\} \longrightarrow^{L^2} F \text{ with } F^n \in K(H^n),\ \forall n \in \mathbb{N},$$

then $F \in K(H)$.
Let $\{H^n\}, \{F^n\} \in L^2$. If $F^n \in K(H^n)$ then it results that:

$$\lambda_r(t) \leqslant F_r^n(t) \leqslant \mu_r(t) \text{ a.e. in } [0,T]\, \forall r = 1, 2\ldots., m$$
$$\{F^n\} \longrightarrow^{L^2} F,$$

and consequently the capacity constraints on $F$ are satisfied. Moreover the following relationship can be easily proved:

$$\sum_{r=1}^{m} \varphi_{jr} F_r(t) = \frac{1}{T} \int_0^T \rho_j(t, H(\tau)) \mathrm{d}\tau \;\text{ a.e. in } [0,T],\; j = 1, 2, \ldots, l$$

and therefore we deduce that $F \in K(H)$.

In order to show the lower semi-continuity of K, we prove that $\forall \{H^n\} \longrightarrow^{L^2} H, \forall F \in K(H)$ there exists $\{F^n\}$ such that:

$$\{F^n\} \longrightarrow^{L^2} F \text{ with } F^n \in K(H^n)\; \forall n \in \mathbb{N}.$$

Let us consider an arbitrary $\{H^n\} \longrightarrow^{L^2} H, F \in K(H)$ and fix $n \in \mathbb{N}, t \in [0,T]$. We introduce the following sets:

$$\begin{aligned}
A_j &= \{r \in \{1, 2, \ldots, m\} : \varphi_{jr} = 1\}\\
B_j(n,t) &= \{r \in A_j : \bar{\rho}_j(t) - \bar{\rho}_j^n(t) \leqslant 0\}\\
C_j(n,t) &= \{r \in A_j : 0 < \bar{\rho}_j(t) - \bar{\rho}_j^n(t) < F_r(t) - \lambda_r(t)\}\\
Dj(n,t) &= \{r \in A_j : F_r(t) - \lambda_r(t) \leqslant \bar{\rho}_j(t) - \bar{\rho}_j^n(t)\}
\end{aligned}$$

where $j \in \{1, 2, \ldots, l\}$ and

$$\bar{\rho}_j(t) = \frac{1}{T} \int_0^T \rho_j(t, H(\tau)) \mathrm{d}\tau,\; \bar{\rho}_j^n(t) = \frac{1}{T} \int_0^T \rho_j(t, H^n(\tau)) \mathrm{d}\tau.$$

Let us construct the following sequence $\{F^n\}$:

$$F_r^n(t) = \begin{cases} F_r(t) & \text{if } r \in B_j \cup D_j \\ F_r(t) - \frac{\bar{\rho}_j(t) - \bar{\rho}_j^n(t)}{\sum\limits_{s \in C_j} \varphi_{js}} & r \in C_j. \end{cases}$$

If $r \in B_j \cup D_j$ then $F_r^n(t) = F_r(t)$, whereas if $r \in C_j$ we have that a.e. in $[0,T]$:

$$\lambda_r(t) < F_r(t) - \frac{\bar{\rho}_j(t) - \bar{\rho}_j^n(t)}{\sum\limits_{s \in C_j} \varphi_{js}} < \mu_r(t),$$

thus the capacity restrictions are satisfied.
It is easy to prove that demand requirements are verified and that $\{F^n\}$ converges in $L^2$ to $F$. $K(H)$ is a closed and convex subset of $E \forall H \in E$ and since the space $E$ is compact, $K(H) \forall H \in E$ is compact too. Therefore, all the hypotheses of Tan's theorem are satisfied and the existence of at least one solution is guaranteed. This theorem establishes that even in the presence of capacity constraints an equilibrium solution can be found.

## 5. An example

Let us consider the network as in the figure, where $N = \{P_1, P_2, P_3, P_4\}$ is the set of nodes and $L = \{(P_1, P_2), (P_1, P_3), (P_2, P_4), (P_3, P_4)\}$ is the set of links. We

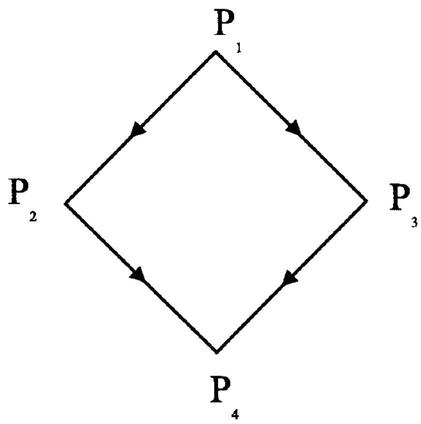


*Figure 1.*

assume that the origin-destination pair is represented by $(P_1, P_4)$, so that the routes are the following:

$$R_1 = P_1P_2 \cup P_2P_4$$
$$R_2 = P_1P_3 \cup P_3P_4.$$

Let us assume that the route costs are the following:

$$C_1(F(t)) = (\beta + 6)F_1(t) + \alpha$$
$$C_2(F(t)) = (\beta + 6)F_2(t) + \alpha$$

where $\alpha, \beta \geqslant 0$.
The set of all the feasible flows is given by:

$$K(H) = \Big\{F \in L^2(0, T; \mathbb{R}^2_+) : \frac{t}{4} + \frac{\varepsilon\delta T}{8 - 4\delta} \leqslant F_i(t) \leqslant t + T \text{ a.e. in } [0, T], \quad i = 1, 2;$$
$$F_1(t) + F_2(t) = \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))d\tau \text{ a.e. in } [0, T]\Big\},$$

where $\frac{1}{2} \leqslant \varepsilon \leqslant 2, 0 \leqslant \delta \leqslant \frac{4}{3}$. The equilibrium flow is the solution of the quasi-variational inequality:

$$H \in K(H), \int_0^T \sum_{r=1}^{2} C_r(H(t))(F_r(t) - H_r(t))\mathrm{d}t \geqslant 0 \qquad \forall F \in K(H). \tag{5.18}$$

Following the procedure shown in [4, 6] we set:

$$\begin{aligned}
F_2(t) &= \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))\mathrm{d}\tau - F_1(t);\\
\widetilde{E} &= \Bigg\{ \widetilde{H} \in L^2(0,T) : \frac{t}{4} + \frac{\varepsilon\delta T}{8-4\delta} \leqslant H_1(t) \leqslant t + T \text{ a.e in } [0,T];\\
&\quad \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))\mathrm{d}\tau - (t+T) \leqslant H_1(t) \leqslant \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))d\tau +\\
&\quad - \left(\frac{t}{4} + \frac{\varepsilon\delta T}{8-4\delta}\right) \text{ a.e in } [0,T] \Bigg\};\\
\widetilde{K}(H) &= \Bigg\{ \widetilde{F} \in L^2(0,T) : \frac{t}{4} + \frac{\varepsilon\delta T}{8-4\delta} \leqslant F_1(t) \leqslant t + T \text{ a.e in } [0,T];\\
&\quad \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))\mathrm{d}\tau - (t+T) \leqslant F_1(t) \leqslant \frac{1}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))\mathrm{d}\tau +\\
&\quad - \left(\frac{t}{4} + \frac{\varepsilon\delta T}{8-4\delta}\right) \text{ a.e in } [0,T] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\Gamma(\widetilde{F}(t), \widetilde{H}(t)) &= C_1(\widetilde{F}(t), \widetilde{H}(t)) - C_2(\widetilde{F}(t), \widetilde{H}(t))\\
&\qquad 2(\beta+6)F_1(t) - \frac{\beta+6}{T}\int_0^T (\varepsilon t + \delta H_1(\tau))d\tau.
\end{aligned}$$

Thus, the problem (5.18) can be written as:

$$\widetilde{H} \in \widetilde{K}(\widetilde{H}), \int_0^T \Gamma(\widetilde{H}(t))(\widetilde{F}(t) - \widetilde{H}(t)\mathrm{d}t \geqslant 0 \qquad \forall \widetilde{F} \in \widetilde{K}(\widetilde{H}). \tag{5.19}$$

It is immediate to show that if $\widetilde{H}$ satisfies the system:

$$\begin{cases} \Gamma(\widetilde{H}, \widetilde{H}) = 0 \\ \widetilde{H} \in \widetilde{K} \end{cases}$$

then it solves (5.19). It results: $H_1(t) = \frac{\varepsilon t}{2} + \frac{\varepsilon\delta T}{8-4\delta}$ and since the flow has to satisfy the constraints of the convex set $\widetilde{K}$, it must be:

$$\frac{t}{4} + \frac{\varepsilon\delta T}{8-4\delta} \leqslant H_1(t) \leqslant t + T$$

$$\frac{1}{T}\int_0^T (\varepsilon t+\delta H_1(\tau))d\tau-(t+T)\leqslant H_1(t)$$
$$\leqslant \frac{1}{T}\int_0^T (\varepsilon t+\delta H_1(\tau))d\tau-\left(\frac{t}{4}+\frac{\varepsilon\delta T}{8-4\delta}\right).$$

The above conditions are always verified if $\frac{1}{2}\leqslant\varepsilon\leqslant 2$ and $0\leqslant\delta\leqslant\frac{4}{3}$.
Thus the equilibrium solution is given by

$$\left(\frac{\varepsilon t}{2}+\frac{\varepsilon\delta T}{8-4\delta},\frac{\varepsilon t}{2}+\frac{\varepsilon\delta T}{8-4\delta}\right).$$

A numerical example can be obtained by choosing: $\varepsilon=1$, $\delta=\frac{4}{3}$, $T=1$ and $\alpha,\beta$ arbitrarily fixed.

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